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Quaternionic complexes

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Each regular or semi-regular integral affine orbit of the Weyl group of $\mathfrak{gl}(2n+2,\mathbb{C})$ invariantly determines a locally exact differential complex on a $4n$ dimensional quaternionic manifold. This gives quaternionic analogues of Dolbeault cohomology on complex manifolds. We compute the index of such complexes in the hyper-Kähler case, showing that quaternionic cohomology is not trivial.

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1. Introduction

One of Roger Penrose's key ideas has been that complex analysis, with its in-built rigidity, should somehow encode the laws of physics. The *Penrose transform* between complex analytic cohomology on $\mathbb{CP}^3$ and zero rest mass free fields and the *non linear graviton* description of self-dual metrics best illustrate the power of this viewpoint. Both have led to interesting mathematics. The Penrose transform can be set up for any semisimple Lie group [6], and the natural higher dimensional analogue of self-duality is a quaternionic manifold [28,4]. In this paper I shall show how to use the Penrose transform on quaternionic manifolds to suggest analogues of the Dolbeault complex in holomorphic geometry, following some ideas of Simon Salamon.

There are two rather complementary points of view to adopt in complex analysis. On the one hand there is the Weierstrass school, which defines holomorphicity in terms of analysis, via the convergence of Taylor series. On the other, associated with Cauchy and Riemann, holomorphicity is defined differentially, via the Cauchy–Riemann equations. In seeking a naive theory of quaternionic analysis, it is well known that the Weierstrass route is uninteresting. For if $q = t + ix + jy + kz$ then each of t, x, y, z is a function of q and not $\overline{q}$—for instance $4t = q - iqi - jqj - kqk$—so any quaternion valued real analytic function on $\mathbb{R}^4$ has a convergent quaternionic power series and vice versa. In other words, Weierstrass quaternion analysis is the same as the study of analytic functions on $\mathbb{R}^4$. This led Fueter, in the 1930's, to propose that quaternionic analysis should depend on the kernel of the Cauchy–Riemann–Fueter operator:

$$\psi \longmapsto \frac{\partial}{\partial \overline{q}}\psi \overset{\text{def}}{=} \left(\frac{\partial}{\partial t} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\psi. \tag{1}$$

Now write $\psi = (\psi_{0'}, \psi_{1'})$, identifying $\mathbb{H} = \mathbb{C}^2$, and represent i, j, k by the usual Pauli matrices. Then (1) may be rewritten in matrix form:

$$\psi \longmapsto \begin{pmatrix} \partial/\partial t + \mathrm{i}\partial/\partial x & \mathrm{i}\partial/\partial y - \partial/\partial z \\ \mathrm{i}\partial/\partial y + \partial/\partial z & \partial/\partial t - \mathrm{i}\partial/\partial x \end{pmatrix} \begin{pmatrix} \psi_{0'} \\ \psi_{1'} \end{pmatrix}. \tag{2}$$

Thus, as Simon Salamon has pointed out, Fueter's operator is in fact the well known Dirac–Weyl operator acting on spinors over $\mathbb{R}^4$. Fueter's quaternionic holomorphic functions amount to massless, right-handed neutrino fields, in physical terms. As such, it is amenable to study by twistor methods, since these interpret such fields as holomorphic sheaf cohomology groups on twistor space. This interpretation is via a construction known as the *Penrose transform*, which is now very well understood, particularly from the point of view of representation theory [6].

Such a study is the purpose of this paper. Quaternionic geometry in four real dimensions is the study of conformal geometry, augmented by anti-self-duality assumptions. This is rather a special case, and anyway already exhaustively studied. So attention will focus on eight and higher real dimensions. Precisely, following ref. [28], a (real) quaternionic manifold M is real $4n$ dimensional manifold with a reduced structure group $\mathrm{Sp}(1)\mathrm{GL}(n,\mathbb{H})$ admitting a torsion free connection ∇. An invariant analogue of the Fueter operator is defined as follows. The reduction of the structure group means that the complexified tangent bundle

$$TM_{\mathbb{C}} \cong H \otimes E \quad \Rightarrow \quad T^*M_{\mathbb{C}} = H^* \otimes E^*, \tag{3}$$

where H and E are complex vector bundles of rank 2 and $2n$, respectively. $L = \wedge^2 H$ is a natural line bundle on such a manifold, an analogue of the bundle of conformal densities in four dimensions. ψ is a section of $H^* \otimes L^*$ and the Dirac–Fueter operator is the composition

$$\mathbf{F} : H^* \otimes L^* \xrightarrow{\nabla} T^*M_{\mathbb{C}} \otimes H^* \otimes L^* \xrightarrow{\wedge} E^* \otimes (L^*)^2.$$

In index notation, which we shall use throughout the paper [27,1],

$$\psi_{A'} \longmapsto \nabla_{A[A'}\psi_{B']}.$$

The weighting of ψ by tensoring in L^* makes the Dirac–Fueter operator an invariant of the factorization (3). Indeed, it is then an invariant of this factorization and the torsion free requirement on ∇ and not otherwise dependent on ∇.

In complex analysis the Cauchy–Riemann operator begins the Dolbeault complex which is locally exact. Holomorphic cohomology is the extent to which it is not globally exact. The main result of this paper is to study a quaternionic analogue of cohomology obtained by similarly extending the Fueter operator to a locally exact complex. We shall do this using certain complexes (due to Bernstein–Gelfand–Gelfand) in representation theory, the Penrose transform and the twistor theory of quaternionic manifolds. We shall see that there is a whole family of related complexes, one for each semiregular orbit of the Weyl group of $\mathrm{sl}(2n+2,\mathbb{C})$, each of which might be a Dolbeault analogue. There is one amusing complication. Most involve second order operators; this occurs even for the complex extending the Fueter operator.

Suppose now that V is a quaternionic bundle on M; thus V is equipped with a self-dual connection. Equivalently, V is the Ward transform of a holomorphic vector bundle on quaternionic twistor space. We may couple the connection to the operators in the complex and so define the quaternionic cohomology of V. The twistor approach makes it possible to compute the index of this cohomology, via the Hirzebruch–Riemann–Roch (HRR) theorem on twistor space. The simplest formulae occur on a hyper-Kähler manifold M, when the index of these complexes, coupled to V is a multiple of $[\mathrm{ch}(V)\ \mathrm{td}(E)][M]$. In other words, the HRR theorem descends to a quaternionic analogue. Salamon has found similar formulae by relating the complexes to a Dirac operator.

One possible application of such an idea is to a quaternionic analogue of geometric quantization. Let G be a complex Lie group and T be a complex torus of G. Then G/T is known to be hyper-Kähler, in several ways, each invariant under the left action of a compact real form of G [22]. It should be possible to obtain hyper-Kähler structures invariant under a non compact real form $G_{\mathbb{R}}$. Then the quaternionic cohomology of the complexes presented here will be representations of $G_{\mathbb{R}}$. The orbits of $G_{\mathbb{R}}$ on G/T are resolutions of nilpotent coadjoint orbits and so there is a chance that these resolutions bear some relation to unipotent representations [30].

All of the constructions in this paper really only depend on the factorization (3) and the requirement that TM admit a torsion free connection; it is unnecessary, for instance, to suppose that a compatible metric is given (i.e., to reduce to $\mathrm{Sp}(n)$ on the second factor). Indeed, for algebraic simplicity it will be easiest to work over $\mathbb{C}$, rather than $\mathbb{R}$. This also allows the Penrose transform to be given in its simplest form (via a *double fibration*). A real quaternionic manifold is analytic, since its twistor space is holomorphic, so it is sufficient to establish the existence and local exactness of quaternionic complexes holomorphically. Actually, as Eastwood has pointed out, once the complexes are established in the analytic category, they hold in the smooth category, by the work of Nacinovich [25].

One feature of this paper is that it relies a good deal on recent work in the theory of Lie algebras, particularly on the structure theory of Verma modules. Readers unfamiliar with this work should consult ref. [6]. Similar methods are used in refs. [4,5].

2. Local twistor theory

We shall rely heavily on the *local twistor theory* [1,4,10,27] of a quaternionic manifold. Reference [4] places quaternionic manifolds in the context of a much wider class of manifolds, modelled locally on Hermitian symmetric spaces. The common thread is the idea of second order structures based on semisimple Lie groups [23,26] and one way to study them is via jet bundles and Cartan normal connections. An alternative, convenient and instructive method is to extend original ideas of Penrose, showing that the existence of a particular kind of G_0-structure $\mathcal{Q} \subset F^1(M)$ on a manifold leads to an uniquely defined connection on an auxiliary bundle. Interesting differential geometry arises when we study the integrability of this connection. Full details are given in ref. [4] and particularly in ref. [1], in index notation. See also ref. [24, p. 51] and refs. [16,17] for similar ideas based on conic structures.

2.1. COMPLEX QUATERNIONIC MANIFOLDS

In the quaternionic case we could set $G_0 = \mathrm{Sp}(1)\mathrm{GL}(n,\mathbf{H})$; but since our considerations are purely local and for algebraic simplicity, we shall instead complexify and let $G_0 = \mathrm{S}(\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2n,\mathbb{C}))$. Thus M is now a complex manifold and we are studying a G_0-structure on $\Theta = T^{1,0}M$, i.e. a $G_0/\mathbb{Z}_2$ principal subbundle $\mathcal{Q}$ of the bundle of holomorphic frames on M. Nothing is lost in doing this and we gain a clearer insight into the underlying representation theory as a result. Reality conditions are easily imposed later on. We adopt the convention that the Lie algebra of a Lie group is denoted by the corresponding bold face lower case letter.

At the level of Lie algebras, $\mathbf{g}_0$ is the reductive Levi factor of a maximal parabolic subalgebra $\mathbf{p} = \mathbf{g}_0 \oplus \mathbf{g}_1 \subset \mathbf{g}$ where $\mathbf{g}_1$ is Abelian and $\mathbf{g} = sl(2n+2,\mathbb{C})$. We can find an Abelian complement $\mathbf{g}_{-1}$ of $\mathbf{p}$ so that $[\mathbf{g}_i, \mathbf{g}_j] \subset \mathbf{g}_{i+j}$ (setting $\mathbf{g}_{\pm 2} = 0$). Then the adjoint representation of $\mathbf{g}_0$ on $\mathbf{g}_{-1}$ is the one through which the structure on Θ is defined. This is simply the tensor product of the self-representations $\mathbb{H}, \mathbb{E}$ of the factors of G_0 and so $\boldsymbol{\Theta}$ is a tensor product

$$\boldsymbol{\Theta} = H \otimes E, \tag{4}$$

modifying the notation of ref. [28] slightly. Thus if $\Omega^1 = T^{*\,1,0}M$ then there is a splitting $\Omega^2 = \Omega^2_+ \oplus \Omega^2_-$, into what may be called self- and anti-self-dual parts, where

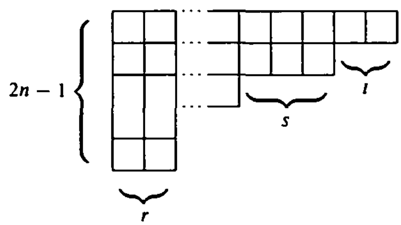
$$\Omega^2_+ = S^2H^* \otimes \wedge^2E^*, \qquad \Omega^2_- = \wedge^2H^* \otimes S^2E^*.$$

The "S" in G_0 means that additionally we are given a fixed isomorphism $\det H \cong \det E$. We shall denote this bundle by $\mathcal{O}[1]$; thus $\mathcal{O}[p] \cong (\det H)^p$.

We shall be interested in other irreducible bundles built up from tensor products of H, E. A handy notation, introduced in ref. [6], specifies a bundle by specifying the lowest weight of an inducing representation. This notation will be crucial below, so we shall briefly review it. Fix a Cartan subalgebra $\mathbf{h} \subset \mathbf{g}$, a system $S = \{\alpha_i\}$ of simple roots and let $\{\lambda_i\}$ be dual the basis of fundamental weights. The weight $\lambda = \sum n_i\lambda_i$ is denoted by writing n_i over the ith node in the Dynkin diagram of $\mathbf{g}$—the second node is crossed through to record the Dynkin diagram of $\mathbf{g}_0$. The same diagram represents the bundle over M induced by the representation of $\mathbf{g}_0$ of lowest weight $-\lambda$ which we also denote by $\mathcal{O}(\lambda)$. Thus

$$\overset{p}{\bullet}\!-\!\overset{q}{\times}\!-\!\overset{r}{\bullet}\cdots\overset{s}{\bullet}\!-\!\overset{t}{\bullet} \;=\; S^pH \otimes Y_{r,\dots,s,t}E \otimes \mathcal{O}[q] \;=\; \mathcal{O}(\lambda),$$

where $Y_{r,\dots,s,t}$ is the Young symmetrizer



Here, $p, q, r, \dots, s, t$ are all integers—over $\mathbb{R}$, q may be any real number, though the complexes below only occur for q integral. It is easy to check that

$$E = \overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\,, \qquad E^* = \overset{0}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{1}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\,,$$

$$H = \overset{1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\,, \qquad E^* = \overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\,,$$

$$\Omega^2_+ = \overset{2}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\,, \qquad \Omega^2_- = \overset{0}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{2}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\,,$$

$$\wedge^2E = \overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\,, \qquad S^jH = \overset{j}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\,,$$

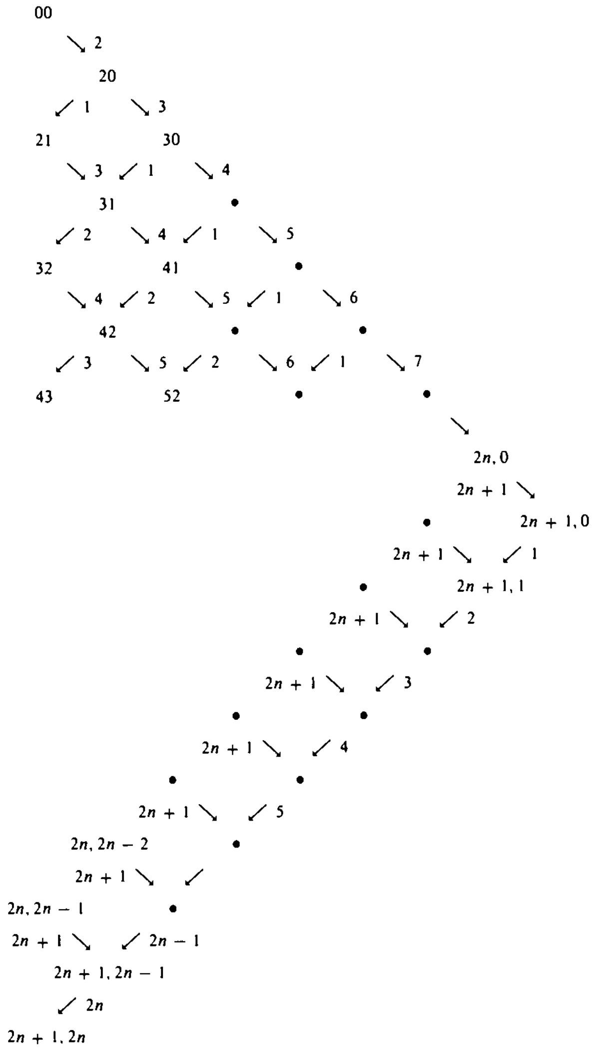


Fig. 1. Quaternionic Hasse graph. ij represents x_{ij}.

and so forth.

The standard flat model for all of this is the Grassmanian $\mathbf{Gr}_2(\mathbb{C}^{2n+2})$; H^* is the tautological bundle and E the quotient bundle. The case $n = 1$ is the Klein quadric and, since $S(GL(2,\mathbb{C}) \times GL(2,\mathbb{C})) \xrightarrow{2:1} CO(4,\mathbb{C})$, the flat model for four-dimensional conformal geometry. This is the original case studied by Penrose; it is slightly special and of course much studied elsewhere so we shall suppose $n \geq 2$ in the sequel. Also it is not really necessary to suppose that E has even rank, until a metric is required.

2.2. HASSE GRAPHS AND DIFFERENTIAL FORMS

In general, Ω^j splits into a direct sum of irreducible components. To keep track of these components we shall use a directed graph called a *Hasse diagram*, which occurs in representation theory. There are several equivalent ways of describing this. For instance, it codes the topological structure of $\mathrm{Gr}_2(\mathbb{C}^{2n+2})$ via a Morse stratification in which each node represents a stratum and each edge a Morse flow between strata. Combinatorially, it is the coset space $W^{\mathfrak{p}} = W_{\mathfrak{g}_0} \setminus W_{\mathfrak{g}}$ where $W_{\mathfrak{g}} = S_{2n+2}$ is the Weyl group of $\mathfrak{g}$ and $W_{\mathfrak{g}_0} = S_2 \times S_{2n}$ is that of $\mathfrak{g}_0$. Recall that this has a natural length functional and Bruhat ordering—each coset has a unique minimal length representative, and so $W^{\mathfrak{p}}$ is a directed graph. If σ_i is the reflection perpendicular to the ith simple root [i.e., the simple transposition $(i, i+1)$] set

$$x_{ij} = \sigma_2\sigma_3 \cdots \sigma_i\sigma_1\sigma_2 \cdots \sigma_j$$

if $1 \leq j < i \leq 2n+1$, with $x_{00} = \mathrm{id}$ and $x_{i0} = \sigma_2\sigma_3 \cdots \sigma_i$, $2 \leq i \leq 2n+1$. Then $W^{\mathfrak{p}} = \{x_{ij}\}$ is given in fig. 1. Directed edges are of the form

$$\begin{array}{ccc} & x_{ij} & \\ {}^{j+1}\swarrow & & \searrow^{i+1} \\ x_{i,j+1} & & x_{i+1,j} \end{array}$$

Thus an arrow labelled i corresponds to right action by σ_i on $W^{\mathfrak{p}}$. The length of x_{ij} is $l(x_{ij}) = i + j - 1$. This is the length of the shortest path from x_{00} to x_{ij}.

We can now say how to compute the irreducible summands of Ω^m. Let ρ be half the sum of the positive roots of $\mathfrak{g}$, and for any $w \in W_{\mathfrak{g}}$ and any weight λ define $w.\lambda = w(\lambda + \rho) - \rho$. In terms of fundamental weights $\sigma_i \sum n_j\lambda_j = \sum n_j\lambda_j + n_i(\lambda_{i-1} - 2\lambda_i + \lambda_{i+1})$. In other words, the jth simple reflection acts on a weight in our notation by adding the coefficient of the jth node to its neighbours and reversing its sign.

Lemma 1.

$$\Omega^m = \bigoplus_{l(x_{ij})=m} \mathcal{O}(x_{ij}.0).$$

Furthermore, the projection of the exterior differential $d : \mathcal{O}(x_{ij}.0) \to \mathcal{O}(x_{kl}.0)$ *is non-zero if and only if* $x_{ij} \to x_{kl}$ *in the Hasse diagram.* □

This is a more or less standard fact in representation theory, where the de Rham complex becomes a resolution of the trivial module due to Bernstein, Gelfand and Gelfand—see ref. [6, ch. 8] for details.

Example 2. To illustrate this we compute the summands of the three-forms for a quaternionic structure in six dimensions. The relevant elements of the Hasse graph in fig. 1 are

$$x_{30} = \sigma_2\sigma_3\sigma_4 \quad \text{and} \quad x_{31} = \sigma_2\sigma_3\sigma_1.$$

Thus

$$x_{30}\rho = \overset{4}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{1}{\bullet} \quad \text{and} \quad x_{30}\rho = \overset{2}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{2}{\bullet}\,,$$

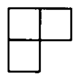
so that

$$x_{30}.\rho = \overset{3}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \quad \text{and} \quad x_{30}.\rho = \overset{1}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{1}{\bullet}\,.$$

Hence

$$\Omega^3 = S^3H^*[-1] + H^* \otimes Y \cdot E[-1],$$

where Y is the Young diagram



2.3. GENERAL RESOLUTIONS

The general Bernstein–Gelfand–Gelfand resolution is a resolution of a finite dimensional g module. If $-\lambda$ is the lowest weight of an irreducible module $\mathbb{E}(\lambda)$ then there is a resolution

$$0 \to \mathbb{E}(\lambda) \to \mathcal{O}(\lambda) \to \mathcal{O}(x_{20}.\lambda) \to \ldots \to \bigoplus_{l(x_{ij})=m} \mathcal{O}(x_{ij}.\lambda)$$
$$\to \ldots \to \mathcal{O}(x_{2n-1,2n-2}.\lambda) \to 0$$

on $\mathbf{Gr}_2(\mathbb{C}^{2n+2})$. Again, non-trivial differentials are given by the edges of the Hasse graph. This is defined on a general M, but not a complex—failure occurs because of curvature. The differentials are invariant differential operators, with the same symbol as in the flat case [3,5]. Similar resolutions occur in the Penrose transform.

2.4. LIE ALGEBRA COHOMOLOGY

We shall be particularly interested in studying bundles over M induced from a representation of G restricted to G_0. Let $\mathbb{F}$ be such a (finite dimensional) representation—the ones of most interest are the adjoint representation $\mathbf{g}$ and the self-representation $\mathbb{T}$. Then $\mathbb{F} = \oplus\mathbb{F}_i$ as a G_0 module and

$$\mathbf{g}_i \cdot \mathbb{F}_j \subset \mathbb{F}_{i+j}. \tag{5}$$

For instance, $\mathbb{T} = \mathbb{E} \oplus \mathbb{H}^* = \mathbb{T}_0 \oplus \mathbb{T}_1$. Let F, F_i be the induced bundles. As Θ, Ω^1 are induced by $\mathbf{g}_{\mp 1}$, respectively, and $\mathbf{g}_{\mp 1}$ are dual, by the Killing form, (5) yields a natural map

$$\partial : F_i \to F_{i-1} \otimes \Omega^1, \qquad \partial f(X) = X \cdot f.$$

Extending ∂ by the usual formula for an exterior differential we obtain a bigraded complex $E^{p,q}(\mathbb{F}) = F_{-q} \otimes \Omega^{p+q}$ with

$$\partial : E_0^{p,q} \to E_0^{p,q+1}.$$

This has a natural adjoint [4,21,26]

$$\partial^* : E_0^{p,q} \to E_0^{p,q-1}$$

and so a "Laplacian" $\Box = \partial^* \partial + \partial \partial^*$ and a natural Hodge structure [21]:

$$\begin{array}{c} E_0^{p,q} = \operatorname{im} \partial \oplus E_1^{p,q} \oplus \operatorname{im} \partial^*. \\ \| \\ \ker \partial \cap \ker \partial^* = \ker \Box. \end{array}$$

In fact, the cohomology $E_1^{p,q}$ of this complex is simply (the bundle induced by) the Lie algebra cohomology $H^*(\mathbf{g}_{-1}, \mathbb{F})$ given by Kostant's formula [21]. See ref. [4, table 1] for details. (Translate into the notation of that paper by observing $E_1^{p,q} = H^{p+q,-q}$. We choose to bigrade E so that ∂ becomes a vertical differential. A flat connection ∇ on Q may be added as a horizontal differential and the resulting double complex is extremely interesting in connection with invariant differential operators [5].)

2.5. THE WEYL STRUCTURE ON M

This is applied as follows. The torsion T of any G_0 connection ∇ lies in $\Theta \otimes \Omega^2 = E^{1,1}(\mathbf{g})$ and so $\partial T = 0$. If $\hat{\nabla} = \nabla + C$ is a second G_0 connection then $C \in E^{1,0}(\mathbf{g})$ and $\hat{T} - T = \partial C$. In particular, if $C = \partial \Upsilon$, $\Upsilon \in \Omega^1$, then $\hat{T} = T$. In the present case, $E_1^{1,0} = 0$ which yields

Lemma 3. *There is a distinguished family* $[\nabla]$ *of* G_0 *connections on* M *for which* $\partial^* T = 0$. *Any two connections in this family differ by* $\partial \Upsilon$ *for some one-form* Υ. *The torsion* T *of any connection in this class depends only on the* G_0 *structure.* □

Otherwise put, M admits a distinguished *Weyl structure*. The torsion condition becomes

(i) if $n = 1$ (four-dimensional conformal geometry) then $T = 0$ and
(ii) if $n \geq 2$ then T is totally trace free, that is, a section of

$$T \in \overset{3}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet} = S^3H^* \otimes \wedge^2 E \subset \Theta \otimes \Omega^2_+.$$

The conformal case $n = 1$ is Penrose's non-linear graviton and is something of a special case. We shall suppose $n \geq 2$. Then M is (complex) quaternionic if and only if its invariant torsion vanishes. Notice how natural such a definition is in the context of Lie algebra cohomology.

2.6. PRACTICAL CALCULATION

For calculation purposes it is often useful to distinguish a special subclass of connections within this Weyl structure [1,4]. Then we may adapt Roger Penrose's two-component spinor notation. If in addition to $\partial^* T = 0$ we require that $\nabla\epsilon = 0$ for a fixed section ϵ of $\wedge^2 H^*$ then ∇ is completely fixed. For instance, a metric compatible with the quaternionic structure on M is a section $g = \epsilon \otimes \omega$ of $\wedge^2 H^* \otimes \wedge^2 E^*$. We may use the given isomorphism $\wedge^2 H^* \cong \wedge^{2n} E^*$ to require also that $\epsilon \cong \omega^n$. This fixes ϵ and the corresponding ∇ is simply the Levi-Civita connection. In the sequel we shall only work with such connections. Since we are concerned only with quaternionic structures, they are torsion free. If $\epsilon \rightsquigarrow \hat{\epsilon} = \kappa\epsilon$ then $\Upsilon = d\log\kappa$.

Following ref. [1, eqs. (13), (14)] and ref. [27, p. 242] decompose the curvature operator $2\Omega = \Omega^+ + \Omega^-$ into self- and anti-self-dual components. In index notation,

$$2\nabla_{[a}\nabla_{b]} = \Box_{AB}\epsilon_{A'B'} + \Box_{A'B'AB}.$$

Then

$$\begin{aligned}
\Box_{AB}\mu^D &= (\Psi^D_{ABC} - 2\Lambda_{C(A}\delta^D_{B)})\mu^C,\\
\Box_{AB}\mu^{D'} &= \Phi^{B'}_{ABA'}\mu^{A'},\\
\Box_{A'B'AB}\mu^D &= 2\delta^B_{[A}\Phi_{B]CA'B'}\mu^C,\\
\Box_{A'B'AB}\mu^{D'} &= -2\Lambda_{AB}\delta^{D'}_{(A'}\mu_{B')}.
\end{aligned}$$

(The reader unfamiliar with Penrose's spinor index notation should consult ref. [27]. These formulae make use of the isomorphism $H \cong H^*$ afforded by ϵ. Thus μ^A and $\mu^{A'}$ are a section of E and H, respectively, and indices are raised and lowered by $\mu_{A'} = \mu^{B'}\epsilon_{B'A'}$, $\mu^{A'} = \epsilon^{A'B'}\mu_{B'}$.)

Each component of the curvature appearing in these formulae is irreducible. Thus

$$\begin{aligned}
\Psi^D_{ABC} &\in \overset{0}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{3}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{1}{\bullet} = (E \otimes S^3E)^o[-1],\\
\Phi_{ABA'B'} &\in \overset{2}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet} = S^2H^* \otimes S^2E^*,\\
\Lambda_{AB} &\in \overset{0}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet} = \wedge^2E^*[-1].
\end{aligned}$$

In particular, Ricci curvature is given by

$$R_{ab} = -2\Phi_{ABA'B'} + 2(n+2)\Lambda_{AB}\epsilon_{A'B'}.$$

Thus

$$\Lambda_{AB} = \frac{1}{4(n+2)} R_{[AB]A'B'}\epsilon^{A'B'}.$$

2.7. LOCAL TWISTORS

The local twistor bundle on M is constructed by pushing out the first jet exact sequence of E:

$$\begin{array}{ccccccccc} & & E \otimes H^* \otimes E^* & & & & & & \\ & & \| & & & & & & \\ 0 \to & & E \otimes \Omega^1 & \to & J^1(E) & \to & E & \to & 0 \\ & \mathrm{tr} = \partial^* & \downarrow & \text{push} & \downarrow & & \| & & \\ 0 \to & & H^* & \to & \mathcal{T} & \to & E & \to & 0. \end{array}$$

Any connection ∇ on E provides a linear splitting $E \to J^1(E)$—that is simply the definition of a connection. Since $\partial : H^* \hookrightarrow E \otimes \Omega^1$, this means that ∇ realizes $\mathcal{T}$ as a subbundle of $J^1(E)$. Equivalently, ∇ defines a first order linear differential equation on E, the *twistor* equation

$$\nabla\omega + \partial\pi = 0. \tag{6}$$

Here ω is a section of E and (6) is to hold for some π of H^*. Equivalently, the totally trace free part $(\nabla\omega)^o = 0$. This is invariant under $\nabla \rightsquigarrow \nabla + \partial\Upsilon$, so $\mathcal{T} \subset J^1(E)$ is uniquely defined by the G_0 structure on M.

Equation (6) is overdetermined and leads to a connection on $\mathcal{T}$: apply ∇, observe $\nabla\partial + \partial\nabla = T$ and use $\partial^* T = 0$ to obtain $\nabla\pi$ in terms of ω and curvature. Restricting to a curve gives a well defined system of ordinary differential equations and so a *local twistor connection* [1]:

$$\nabla\begin{pmatrix}\omega \\ \pi\end{pmatrix} = \begin{pmatrix}\nabla\omega + \partial\pi \\ \nabla\pi + P\cdot\omega\end{pmatrix}$$

Here, $P = -\Box^{-1}\partial^*\Omega$ $(\Omega = \nabla^2)$ is a section of $E^{2,-1}(\mathbf{g}) = \Omega^1 \otimes \Omega^1$, the first factor acting via $\mathbf{g}_1 : \mathbb{H} \to \mathbb{E}$. ∂^* effects a trace and $\Box^{-1}$ acts by scalars, so P

is a reweighted linear combination of the irreducible components of the Ricci tensor [1]:

$$P_{ab} = \Phi_{ABA'B'} - \Lambda_{AB}\epsilon_{A'B'}.$$

A straightforward calculation proves the following [1,4]

Lemma 4. ∇ *depends only on the distinguished class* $[\nabla]$ *on* M. □

As an $\mathrm{SL}(2n+2,\mathbb{C})$ invariant connection, $\boldsymbol{\nabla}$ has curvature

$$\boldsymbol{\Omega} = \begin{pmatrix} \Omega + \partial P & T \\ T \cdot P + \nabla P & \Omega + \partial P \end{pmatrix}.$$

Lemma 5. *If the invariant torsion* T *vanishes then* $\boldsymbol{\Omega}$ *is anti-self-dual and takes values in* $\mathrm{End}(E) \oplus \mathfrak{g}_1$.

Proof. $T = 0$ and the structure equation $\partial\Omega = \nabla T$ [4, eq. (16)] gives

$$\partial(\Omega + \partial P) = 0,$$

whilst the definition of P gives

$$\partial^*(\Omega + \partial P) = 0,$$

so that $\Psi = \Omega + \partial P$ is a section of $E_1^{2,0}(\mathfrak{g}) = (\mathrm{End}(E) \otimes \Omega_-^2)^o$. By the Bianchi identity, and the vanishing of torsion

$$\partial\nabla P = -\nabla\Psi.$$

On the other hand, ∂ intertwines the action of G_0, and it is a straightforward exercise to show that the G_0-types in the self-dual part of ∇P do not occur in $\nabla\Psi$. But $E_1^{3,-1} = 0$, so ∂ is injective on ∇P. □

Ψ is therefore an invariant of the quaternionic structure, playing the role of anti-self-dual Weyl curvature in four dimensional conformal geometry.

2.8. TWISTOR SPACES

Let $\mathcal{G} \subset F(\mathcal{T})$ consist of those (unit volume) frames of $\mathcal{T}$ which extend frames of E^*, so $\mathcal{G}$ is a P-principal bundle over M. Since such a frame gives frames of H, E there is a natural map $\phi : \mathcal{G} \to \mathcal{Q} \subset F^1(M)$. The connection ∇ on $F(\mathcal{T})$ provides a horizontal subspace in $T\mathcal{G}$ which is canonically isomorphic to $\mathbf{g}_{-1}$ via $d\phi$. Thus we have an isomorphism

$$\omega : T\mathcal{G}_g \cong \mathbf{g}$$

which is Ad-invariant under P and consistent with the identification of $\mathbf{p}$ with vertical vector fields on $\mathcal{G}$. Such a structure is called a *Cartan connection* [20]. The point of this construction is that Ω is the curvature of ω and so is the extent to which

$$\omega^{-1} : \mathbf{g} \to \Gamma(\mathcal{G}, T\mathcal{G})$$

is *not* a homomorphism of Lie algebras. In the flat case, $M = \mathbf{Gr}_2(\mathbb{C}^{2n+2})$, $\mathcal{G} = G$ and ω is the (left invariant) Maurer–Cartan form.

Let $\mathbf{r} \subset \mathbf{g}$ be the parabolic subalgebra with Levi factor $\mathfrak{gl}(n+1)$, so $G/R = \mathbb{CP}^{n+1}$. Then $\wedge^2(\mathbf{g}_{-1} \cap \mathbf{r})^*$ induces Ω^2_+ and (5) yields that $\omega^{-1}_{|\mathbf{r}}$ is a homomorphism of Lie algebras. This gives a foliation of $\mathcal{G}$. The quotient space by this foliation is the *twistor space* of M [28]. By making suitable convexity assumptions on M we can ensure that Z is a Hausdorff complex manifold. Our considerations will be purely local, so we can and shall assume that such conditions are met. They certainly will be if M is a small complex thickening of a real quaternionic manifold (which is necessarily analytic). Observing that $\mathcal{G}/P \cap R = \mathbf{P}(H)$ yields

Proposition 6. *A (suitably convex, complex) quaternionic manifold M has a twistor space Z related to M by a double fibration*

$$\begin{array}{ccc} & \mathbf{P}(H) & \\ {}^{\eta}\swarrow & & \searrow^{\tau} \\ Z & & M. \end{array}$$ □

Our aim is to use this as the basis of a Penrose transform, from which we shall deduce the quaternionic complexes of interest. Since we are working locally, we shall assume that M is Stein and the fibres of η are contractible [8], [6, section 7.1.2].

We shall use the notation $\mathcal{O}_\mathbf{p}, \mathcal{O}_\mathbf{q}$ or $\mathcal{O}_\mathbf{r}$ to indicate induced bundles on M, $\mathbf{P}(H)$ and Z, respectively.

2.9. THE PENROSE TRANSFORM

The Penrose transform is explained fully in ref. [6, ch. 7–9]—see also refs. [13,11]. It is a machine which identifies holomorphic cohomology on Z with solutions of differential equations on M. In the flat case, the calculation is made easier by observing that the bundles of interest on $Z \subset \mathbb{CP}^{2n+1}$ are homogeneous, and then using of techniques of representation theory. In fact, the same kind of construction works in the curved setting, precisely because the fibres of η, τ retain their homogeneous structure. There are several ways of looking at this. The simplest conceptually is to consider "$Z = \mathcal{G}/R$" and define homogeneous bundles on Z to be bundles induced from representations of R—the fact that only some neighbourhood of the identity in R acts on a particular leaf in $\mathcal{G}$ is immaterial. It follows that all homogeneous bundles on $\mathbb{CP}^{2n+1}$ have analogues on Z. Where necessary, these coincide with intrinsically defined bundles since

Lemma 7. *The holomorphic tangent bundle Θ_Z on Z is the homogeneous bundle induced by* $\mathbf{g}/\mathbf{r}$.

Proof. The Cartan connection ω identifies $T\mathcal{G}_g \cong \mathbf{g}$, so as a vector space, $\Theta_{gR} \cong \mathbf{g}/\mathbf{r}$. If $x \in \mathbf{g}$, let $x^* = \omega^{-1}x$. Then $[x^*, y^*] - [x, y]^* = \boldsymbol{\Omega}(x^*, y^*)$ takes its values in the distribution $\omega^{-1}\mathbf{r}$, by (5). In particular, the action of $\mathbf{r}$ on $\mathbf{g}/\mathbf{r}$ via the Cartan connection coincides with its adjoint action. □

In particular, there are natural analogues of the Hopf sheaves $\mathcal{O}(k)$ and $\Omega_Z^{2n+1} \cong \mathcal{O}_Z(-2n-2)$ as on projective space.

The key observation [2,12] is now that the full machinery of ref. [6], including the use of relative Bernstein–Gelfand–Gelfand resolutions, applies in computing the cohomology of these sheaves. The quaternionic complexes of ref. [29] appear naturally in the spectral sequences computing $H^*(Z, \mathcal{O}(l))$.

Fix $m \in M$ and let $L_m \subset Z$ be the corresponding projective line. Then restriction of cohomology to the formal neighbourhood of L_m corresponds under the Penrose transform to restricting fields on M to their Taylor series about M [6, p. 184]. It follows that such a restriction is injective, whenever the Penrose transform is valid. This gives a key vanishing

Lemma 8. *If M is Stein then $H^l(Z, \mathcal{F}) = 0$ for $l > 1$ and any homogeneous sheaf $\mathcal{F}$.* □

3. Quaternionic complexes

Perhaps the most basic differential operators on a quaternionic manifold, after the twistor operator, are the Dirac operators. The first of these is defined by

$$\begin{array}{cccc} \mathbf{D}: & \overset{0}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet} & \to & \overset{1}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \\ & \| & & \| \\ & E^*[-1] & & H^*\otimes\wedge^2E^*[-1] \\ & \psi_A & \longmapsto & \nabla_{A'[A}\psi_{B]} \end{array}$$

which is readily seen to be a differential invariant of the quaternionic structure. There is a natural extension of $\mathbf{D}$. Define

$$\begin{array}{rcl} \mathbf{D}: S^pH^*\otimes\wedge^{p+1}E^*[-1] & \to & S^{p+1}H^*\otimes\wedge^{p+2}E^*[-1] \\ \psi_{A\ldots BCA'\ldots B'} & \longmapsto & \nabla_{(A'|[A}\psi_{B\ldots CD]|B'\ldots C')}. \end{array}$$

We shall see in a moment that this is does indeed give a complex (a fact equivalent to the quaternionic (torsion vanishing) condition), which is the first and simplest of an entire series of complexes on a quaternionic manifold. It is exact, except at the left, where the kernel of the Dirac operator will be identified with first cohomology on twistor space.

The second invariant Dirac operator is the Dirac–Fueter operator. Here the rôles of E, H are interchanged:

$$\begin{array}{cccc} \mathbf{F}: & \overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet} & \to & \overset{0}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \\ & \| & & \| \\ & H^*[-1] & & E^*[-2] \\ & \psi_{A'} & \longmapsto & \nabla_A^{A'}\psi_{A'}. \end{array}$$

This may at first sight seem rather more simple than the first. Indeed, as explained in the introduction, it appears in the original work of Fueter where it is thought of as a quaternionic Cauchy–Riemann operator. On the other hand, there is no immediately obvious way of continuing $\mathbf{F}$ into a complex, by first order operators. There is, however, a natural invariant second order operator

$$\begin{array}{cccc} \varDelta: & \overset{0}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet} & \to & \overset{0}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\cdots\overset{0}{\bullet} \\ & \| & & \| \\ & E^*[-2] & & \wedge^3E^*[-3] \end{array}$$

given by

$$\begin{aligned} \psi_A \longmapsto\ & \nabla_{A'[A}\nabla_B^{A'}\psi_{C]} + 4\varLambda_{[AB}\psi_{C]} \\ =\ & \nabla_{A'[A}\nabla_B^{A'}\psi_{C]} + \frac{1}{n+2}\epsilon^{A'B'}R_{A'B'[AB}\psi_{C]}. \end{aligned}$$

This is a simple generalization of the conformally invariant Laplacian $\Delta = \nabla^a\nabla_a + \frac{1}{6}R$ when $n = 1$, as we shall see in a moment. Coupling this to operators

$$\begin{array}{cccc} \mathbf{D}: S^pH^* \otimes \wedge^{p+3}E^*[-3] & \to & S^{p+1}H^* \otimes \wedge^{p+4}E^*[-3] \\ \psi_{A\ldots BCA'\ldots B'} & \longmapsto & \nabla_{(A'|[A}\psi_{B\ldots CD]|B'\ldots C')} \end{array}$$

again yields a complex, exact except at the left, when M is quaternionic.

3.1. TWO SINGULAR COMPLEXES

The first complex is obtained by computing the Penrose transform of $\mathcal{O}_Z(-1) = \mathcal{O}_\mathbf{r}(-\lambda_1)$, following the method in ref. [6, ch. 9]. Here

$$-\lambda_1 = \overset{-1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\ .$$

Recall that this first involves writing down a relative Bernstein–Gelfand–Gelfand resolution:

$$\begin{aligned} 0 &\to \eta^{-1}\mathcal{O}_\mathbf{r}(-\lambda_1) \to \mathcal{O}_\mathbf{q}(-\lambda_1) \to \mathcal{O}_\mathbf{q}(-Z_2.\lambda_1) \\ &\to \cdots \to \mathcal{O}_\mathbf{q}(-Z_{2n+1}.\lambda_1) \to 0. \end{aligned}$$

In this, $Z_l = \sigma_2\sigma_3\cdots\sigma_l$. Next we compute the direct images of each resolvent along the $\mathbf{P}^1$ fibres of $\tau : \mathbf{P}(H) \to M$. For instance, $\mathcal{O}_\mathbf{r}(-\lambda_1)$ restricts to the standard $\mathcal{O}(-1)$ bundle along a τ fibre so its direct images vanish in both degrees. On the other hand, each of $-Z_l.\lambda_1$ is already $\mathbf{p}$-dominant, so has $\mathcal{O}_\mathbf{p}(-Z_l.\lambda_1)$ as its non-trivial direct image, in degree zero. Thus the hypercohomology spectral sequence computing $H^*(Z, \mathcal{O}_Z(-1))$ degenerates to a single lowest row:

$$\begin{array}{|lllll} 0 & 0 & 0 & \ldots & 0 \\ 0 & \mathcal{O}_\mathbf{p}(-Z_2.\lambda) & \mathcal{O}_\mathbf{p}(-Z_3.\lambda) & \ldots & \mathcal{O}_\mathbf{p}(-Z_{2n+1}.\lambda) \\ \hline \end{array}$$

By the vanishing lemma 8 this row is a resolution of $H^1(Z, \mathcal{O}_Z(-1))$. Computing each $Z_l.\lambda$ shows that this is simply the complex begining with the Dirac operator, given at the start of this section. In particular, the cohomology group is identified with the kernel of the Dirac operator D.

To see how the *Fueter–Dirac* operator is extended to a resolution, we compute $H^1(Z, \mathcal{O}_Z(-3))$. This yields a spectral sequence whose E_1 term is

$$\begin{array}{|lllllll} \mathcal{O}_\mathbf{p}(-x_{20}.\lambda_3) & \mathcal{O}_\mathbf{p}(-x_{21}.\lambda_3) & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 \cdot \mathcal{O}_\mathbf{p}(-x_{42}.\lambda_3) & & \mathcal{O}_\mathbf{p}(-x_{52}.\lambda_3) & \ldots & \mathcal{O}_\mathbf{p}(-x_{2n+1}.\lambda_3) \\ \hline \end{array}$$

where

$$\lambda_3 = \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$$

is the third fundamental weight. Deriving once yields a second order operator

$$\Delta : \mathcal{O}_{\mathbf{p}}(-x_{21}.\lambda_3) \to \mathcal{O}_{\mathbf{p}}(-x_{42}.\lambda_3),$$

which, again by the vanishing lemma, completes a resolution, of $H^1(Z, \mathcal{O}_Z(-3))$. This is the second one above, begining with the Dirac–Fueter operator. Notice that the appearance of a second order operator in consequence of the need to derive the spectral sequence twice before it converges.

The formula for Δ is obtained by using Čech representatives of relative cohomology classes. The calculation is an easy variant of that of the Laplacian given in ref. [6, section 9.2]. In this a trivialization of H is chosen; the only complication in the present case is that we cannot choose this to be covariant constant under ∇. This gives rise to the curvature term.

Remark 9. In the hyper-Kähler case, we may choose ∇ to be flat on H. Then H may be locally trivialized by two covariantly constant sections and so the calculation of d_2 operators in the Penrose transform is exactly the same as in the flat case. Indeed, the Ricci curvature of ∇ vanishes, so all the invariant operators which occur take exactly the same form as in the flat case, when expressed in terms of ∇.

3.2. FURTHER SINGULAR COMPLEXES

By considering the Penrose transform of $\mathcal{O}_Z(-k)$ for $1 \le k$ we obtain a whole sequence of locally exact complexes on a quaternionic manifold.

Thus extend the Dirac–Fueter operator to

$$F : S^{k-j-2}H^* \otimes \wedge^j E^*[-1-j] \longrightarrow S^{k-j-3}H^* \otimes \wedge^{j+1}E^*[-k]$$

and Δ to

$$\Delta : \wedge^{k-2}E^*[1-k] \longrightarrow \wedge^k E^*[-k]$$

by the formula

$$\Delta : \psi_{AB\ldots C} \to \nabla_{A'[A}\nabla_B^{A'}\psi_{CD\ldots E]} + 4\Lambda_{[AB}\psi_{CD]\ldots E}.$$

Proposition 10. *Let* $1 \le k \le 2n+1$. *Then*

$$\begin{aligned} B_k \ : \ 0 \ &\to S^{k-2}H^*[-1] \xrightarrow{F} \cdots \xrightarrow{F} \wedge^{k-2}E^*[1-k] \xrightarrow{\Delta} \wedge^k E^*[-k] \\ &\xrightarrow{D} \cdots \xrightarrow{D} S^{2n-k}H^*[-k-1] \to 0 \end{aligned}$$

is exact, except at the left, where it resolves $H^1(Z, \mathcal{O}_Z(-k))$.

Proof. This follows from the E_1 term in the Penrose transform for $\mathcal{O}_Z(-k)$ which is given by

$$\left|\begin{array}{cccccccccc} x_{k-1,0} & x_{k-1,1} & \cdots & x_{k-1,k-2} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & x_{k+1,k-2} & x_{k+2,k-2} & \cdots & x_{2n+1,k-2} \\ \hline \end{array}\right.$$

where we have written x for $\mathcal{O}_{\mathfrak{p}}(-x.\lambda_k)$, to save space. Elementary calculations confirm that these are the bundles in the given sequence and the single non-trivial d_2 operator in E_2, namely Δ, is obtained as before. The result is a resolution, by the vanishing lemma 8. □

Notice, for instance, that the complex for $k = 2$ begins with a second order operator. This is a direct generalization of the conformally invariant Laplacian in four dimensions, which is the case $n = 1$. Thus

$$H^2(Z, \mathcal{O}_Z(-2)) \cong \{\phi \in \mathcal{O}[-1] \mid (\nabla_{A'[A}\nabla^{A'}_{B]} + 4\Lambda_{AB})\phi = 0\}.$$

All these complexes, with $1 \leq k \leq 2n + 1$, are associated to singular weights for $\mathfrak{g}$. That is, each bundle occurring in the complex is indexed by a singular weight λ in the $W_{\mathfrak{g}}$ orbit of a singular weight, $-\lambda_k$. Recall that a weight is singular if $\lambda + \rho$ is orthogonal to some root of $\mathfrak{g}$. Any singular weight is uniquely conjugate under the affine action of $W_{\mathfrak{g}}$ to some singular λ with $\lambda + \rho$ in the closure of the dominant Weyl chamber, i.e. $\langle \lambda + \rho, \alpha \rangle \geq 0$ for every positive root α. Equivalently, if

$$\lambda = \overset{p}{\bullet}\!\!-\!\!\overset{q}{\bullet}\!\!-\!\!\overset{r}{\bullet} \cdots \overset{s}{\bullet}\!\!-\!\!\overset{t}{\bullet}$$

then $p, \ldots, t \geq -1$. If exactly one of these, say the kth, is -1 then λ is called *semiregular* for the simple root α_k as are the weights in its affine orbit under $W_{\mathfrak{g}}$. The only weights in this orbit corresponding to homogeneous bundles on M are those in $W^{\mathfrak{p}}.\lambda$ and on Z are those in $W^{\mathfrak{q}}.\lambda$, which are $\mathfrak{p}$- and $\mathfrak{q}$-dominant, respectively.

There is a useful (and standard) algorithm [15,9] for determining which these are. Suppose $\lambda + \rho$ is orthogonal only to α_k. (Thus the integer over the kth node of λ is -1.) Then a weight $x.\lambda$ is of the desired form if and only if one of the edges incident on $x \in W^{\mathfrak{p}}$ corresponds to left multiplication by σ_k. Thus an edge labelled k is attached to x. If so, $x\sigma_k.\lambda = x.\lambda$.

Of course, a similar result holds for $\mathfrak{q}$. But the Hasse graph $W^{\mathfrak{q}}$ is particularly simple, since it corresponds to a cell decomposition of some complex projective space:

$$W^{\mathfrak{q}} = \{\mathrm{id}, z_1 = \sigma_1, \ldots, z_l = \sigma_1\sigma_2\cdots\sigma_l, \ldots, z_{2n+1}\}.$$

It follows that there is a unique homogeneous bundle $\mathcal{O}_{\mathfrak{r}}(z_k.\lambda)$ on Z for each semiregular orbit. It is evident that the calculation of the Penrose transform depends only on k, and not on the particular semiregular orbit associated to α_k. To be more precise, the two terms E_1, E_2 in the Penrose transform for $\mathcal{O}_{\mathfrak{r}}(z_k.\lambda)$ will have the same form as those for $\mathcal{O}(-k)$, with $-\lambda_k$ replaced by λ. One word of caution: the differential operators *and* their degrees will change. Thus the Penrose transform yields an entire cone of invariant resolutions locally on any quaternionic manifold.

Theorem 11. *Let λ be weight for $\mathfrak{g}$ satisfying*

$$\langle \lambda, \alpha_i \rangle \begin{cases} = -1 & \textit{if } i = k, \\ \geq 0 & \textit{otherwise.} \end{cases}$$

Then on a (suitably convex, complex) quaternionic manifold M (associated to $\mathfrak{g}$) with twistor space Z there is an exact sequence

$$\begin{aligned} 0 &\to H^1(Z, \mathcal{O}_{\mathfrak{q}}(z_k.\lambda)) \to \mathcal{O}_{\mathfrak{p}}(x_{k-1,0}.\lambda) \to \mathcal{O}_{\mathfrak{p}}(x_{k-1,1}.\lambda) \to \cdots \\ &\to \mathcal{O}_{\mathfrak{p}}(x_{k-1,k-2}.\lambda) \xrightarrow{\Delta} \mathcal{O}_{\mathfrak{p}}(x_{k+1,k-2}.\lambda) \to \mathcal{O}_{\mathfrak{p}}(x_{k+2,k-2}.\lambda) \to \cdots \\ &\to \mathcal{O}_{\mathfrak{p}}(x_{2n+1,k-2}.\lambda) \to 0. \end{aligned}$$

□

Remark 12. There is an important distinction between Δ and the other operators in this complex. Δ is an example of a *non-standard* invariant operator and corresponds, in the case M is flat, to a particularly interesting homomorphism of (generalized) Verma modules. The theory of Verma modules and their homomorphisms underlies much of what is going on in the present paper. The relation between this theory and invariant differential operators is well known in representation theory. Geometers may consult refs. [3,4,14]. In particular, Enright and Shelton [15] have made a very full study of semiregular categories of Verma modules in the Hermitian symmetric case. In the quaternionic case their results say that the semiregular Verma modules over $\mathfrak{p}$ form a category equivalent to the regular *projective* category for $\mathbb{CP}^{2n}$. This is reflected, for instance, by the fact that we have obtained complexes with a single irreducible in each degree, up to $2n$.

Remark 13. No interesting complexes arise by considering more singular bundles since any (integral) singular bundle on Z is semiregular.

3.3. REGULAR COMPLEXES

The Penrose transform of the canonical bundle $\mathcal{O}_Z(-2n-2)$, i.e. of $k = 2n+2$ above, has an E_1 term non-trivial only in the first row. Thus again one obtains an invariantly defined complex, resolving $H^1(Z, \mathcal{O}_Z(-2n-2))$ and involving only first order operators—it is given by setting $k = 2n + 2$ in the complex of proposition 10.

Now $\mathcal{O}_Z(-2n-2) = \mathcal{O}_{\mathfrak{q}}(z_{2n+1}.0)$ and so is associated to a regular affine orbit in the space of weights—$0 + \rho$ lies within the dominant Weyl chamber. As far as the Penrose transform is concerned, it is the appearance of "z_{2n+1}" in this which counts. If λ is any dominant regular (integral) weight then the Penrose transform of $\mathcal{O}_{\mathfrak{q}}(z_{2n+1}.\lambda)$ contains nontrivial entries in the first row only. Application of the vanishing theorem 8 again yields a resolution.

Theorem 14. *If λ is a dominant weight for* $\mathfrak{g}$ *and M, Z are as in theorem 11 then there is an exact sequence*

$$0 \to H^1(Z, \mathcal{O}_{\mathfrak{q}}(z_{2n+1}.\lambda)) \to \mathcal{O}_{\mathfrak{p}}(x_{2n+1,0}.\lambda) \to \mathcal{O}_{\mathfrak{p}}(x_{2n+1,1}.\lambda)$$
$$\to \cdots \to \mathcal{O}_{\mathfrak{p}}(x_{2n+1,2n}.\lambda) \to 0. \qquad \square$$

In other words, each regular affine orbit of $W_{\mathfrak{g}}$ leads also to a resolution on quaternionic manifolds but of length $2n + 1$.

Remark 15. One can of course compute the Penrose transform of any $\mathcal{O}_{\mathfrak{q}}(z_k.\lambda)$ for λ dominant. If $k < 2n + 1$ this will not lead to a resolution, as we readily see from the E_1 term, which takes the form

$$\left|\begin{array}{cccccccc} x_{k,0} & x_{k,1} & \cdots & x_{k,k-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & x_{k+1,k} & x_{k+2,k} & \cdots & x_{2n+1,k} \end{array}\right. \qquad (7)$$

(employing the convention that x stands for $\mathcal{O}_{\mathfrak{p}}(x.\lambda)$ again). If M is quaternionically flat, the two possible non-zero d_2 operators both give (non-standard) invariant differential operators. It is *not* clear what happens in general in the curved case, even for $n = 1$.

There is actually a lot of interest in this situation. It is known that each standard invariant differential on a flat almost Hermitian symmetric manifold admits a curved analogue which has the same top order symbol and is obtained by adding curvature correction terms [4]. (In our context, the standard operators are those appearing in the E_1 term of the Penrose transform.) This is now known to be *false* for at least one non-standard operator, namely $(\nabla_a \nabla^a)^3$ on $\mathcal{O}[1]$ over a four dimensional conformal manifold [18]. In the flat case, this operator is obtained from the Penrose transform of $\mathcal{O}_{\mathfrak{r}}(z_1.\lambda_2)$, which is a particular case of (7). So it is reasonable to expect that further study of these regular transforms should throw light on the question of curved analogues of non-standard invariant operators.

4. Index of quaternionic complexes

We now return to the real setting and suppose that M is a compact quaternionic manifold. Then M is necessarily analytic, which follows from the existence of its twistor space Z, and has a Stein neighbourhood in the complete family of lines in Z generated by the fibres of $\pi : Z \to M$, which is complex quaternionic.

A complex vector bundle V on M will be called a quaternionic bundle if it is equipped with a connection whose curvature is self-dual, i.e. a section of

$$S^2E[-1]\mathrm{End}(V) \subset \wedge^2 T^*M \otimes \mathrm{End}(V).$$

V is thus the Ward transform of a holomorphic vector bundle $\mathcal{V}$ on Z, trivial on fibres of π and furthermore the connection on V may be coupled to each of the quaternionic complexes given above. Note that V is analytic and extends to a (sufficiently small) Stein neighbourhood of M as a holomorphic bundle, which has a holomorphic self-dual connection.

This gives a variety of possible definitions of what one might mean by quaternionic cohomology on M. The first step is to relate this to cohomology on Z. For simplicity, we will restrict atention to the complexes of proposition 10, coupled to V. The Penrose transform, applied to an appropriate limit of Stein neighbourhoods of M, then yields:

Lemma 16. $H^i(\Gamma(M, B_k \otimes V)) = H^{i+1}(Z, \mathcal{V} \otimes \mathcal{O}(k))$. □

In particular, each cohomology group is finite dimensional. We could suppose that M is quaternionic Kähler and that V is Hermitian, in which case the complexes are elliptic—the adjoint of $B_k \otimes V$ is simply $B_{2n+2-k} \otimes V$—and the cohomology groups are finite dimensional in the usual way. In any event, the index of $B_k \otimes V$ is well defined on M and

$$\begin{aligned} \mathrm{index} B_k \otimes V &= -\mathrm{index} H^{i+1}(Z, \mathcal{V} \otimes \mathcal{O}(k)) \\ &= -[\mathrm{td}(\Theta_Z).\mathrm{ch}\mathcal{V} \otimes \mathcal{O}(k)][Z] \end{aligned}$$

by the Hirzebruch–Riemann–Roch formula.

To reduce this to an expresion evaluated on M recall that

$$H^*(Z, \mathbb{Z}) = H^*(M, \mathbb{Z})[z]/\langle z^2 + c_1(H)z + c_2(H)\rangle,$$

so we should evaluate the coefficient of z in $\mathrm{td}(\Theta_Z)$ (when reduced by the relation). The simplest situation occurs when $c_1(H) = c_2(H) = 0$; for instance, H may be trivial, as in the hyper-Kähler or hypercomplex cases. (In the Kähler case, at least $c_2(H) = 0$ since det $H = 0$.)

The pull back $\pi^* E$ is holomorphic and

$$0 \to \mathcal{O}(2) \to \Theta_Z \to \pi^* E \otimes \mathcal{O}(1) \to 0$$

is exact: $\pi^* E \otimes \mathcal{O}(1)$ is the normal bundle to the fibres of π.

Consider the hyper-Kähler case. Then E is an $\mathrm{Sp}(n)$ bundle (induced by the self-representation) and weights of $\mathrm{sp}(n)$ give rise to characteristic classes on M. Label the weights of the self-representation as

$$x_1, x_2, \ldots, x_n, -x_n, \ldots, -x_2, -x_1.$$

(That both $\pm x_i$ appear is a consequence of the self-duality of the representation.) These give characteristic classes on M, which we label similarly. Define a formal polynomial on characteristic classes by

$$\mathrm{t}(y) = y/(1 - \mathrm{e}^{-y}).$$

Then the multiplicative property of the Todd genus td means that

$$\mathrm{td}(\Theta_Z) = t(2z) \prod_{i=1}^{n} t(z + x_i) t(z - x_i).$$

Integration of a cohomology class on Z over the fibres of π corresponds to finding the coefficient of z in that class. Accordingly, we should compute the Taylor expansion of the above formula. We write

$$\frac{\mathrm{d}}{\mathrm{d}z} \prod_{i=1}^{n} t(z + x_i) t(z - x_i) \Big|_{z=0} = K \prod_{i=1}^{n} t(x_i) t(x_i) = K \mathrm{td}(E),$$

where

$$K = \sum_{1}^{n} \left[t'(x_i)/t(x_i) + t'(-x_i)/t(-x_i) \right].$$

What makes the hyper-Kähler case easily computable is that quite remarkably each summand in K simplifies to 1 so that

$$\mathrm{td}(\Theta_Z) = (1 + z)(1 + nz)\,\mathrm{td}(E) = [1 + (n + 1)z]\,\mathrm{td}(E).$$

This proves

Proposition 17. *If M is hyper-Kähler and V is quaternionic then*

$$\mathrm{index}\, B_k(V) = (k - n - 1)\,\mathrm{ch}(V)\,\mathrm{td}(E)[M]. \qquad \square$$

As a check, if $k = n+1$ then index $B_k(V) = 0$, as it should since the complex $B_{n+1}(V)$ is self-adjoint.

Lemma 18. *If M is a 4n dimensional irreducible compact hyper-Kähler manifold [so its holonomy is precisely* $\mathrm{Sp}(n)$*] then* $\mathrm{td}(E)[M] = n+1$. □

This is well known [7], for a hyper-Kähler structure is equivalent to the existence of a non-degenerate holomorphic two-form ω (i.e., a skew form on E which with a skew form on H gives the complexification of the metric). Powers of this form give sections of Ω^{2r} for each r. Irreducibility implies these are all such sections, up to scale. For the trivial representation occurs exactly once in each even skew power of the self-representation of $\mathrm{Sp}(n)$, and never in the odd powers, whilst any holomorphic form is covariant constant. Thus

$$H^0(M, \Omega^{2r}) = \mathbb{C} \quad \text{and} \quad H^0(M, \Omega^{2r+1}) = 0,$$

which gives the lemma, since M is Kähler.

Thus we deduce:

Proposition 19. *If M is a 4n dimensional irreducible compact hyper-Kähler manifold then*

$$\mathrm{index}\, B_k = (k-n-1)(n+1). \qquad \square$$

So quaternionic cohomology is non-trivial.

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